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An Inverse Problem in Boundary-Layer Flows: Numerical Determination of Pressure Gradient for a Given Wall Shear*

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The problem of determining a pressure gradient distribution that will produce a specified shear force on a body surface in boundary-layer flows is considered. This leads to an "overdetermined" boundary value problem for a partial differential equation containing an unknown coefficient. A numerical procedure for determining the coefficient is given along with several worked out examples including both similar and nonsimilar flows. The method essentially treats the unknown coefficient as an eigenvalue which is computed using Newton's method. This in turn employes a very accurate and efficient finite difference scheme for computing standard boundary-layer flows. Richardson extrapolation is applicable but only modest improvement was obtained in the present examples (for reasons that are explained).

1. INTRODUCTION

In typical boundary-layer flow problems a pressure distribution over some body surface is prescribed and then, by solving the boundary-layer equations, the shear forces on the body can be obtained. However, for various purposes, including the design of optimal body shapes, it is desired to prescribe the shear forces on the body and to determine the pressure distribution which will yield them. Mathematically, this leads to a form of inverse problem in which a coefficient (scalar or function) in an (ordinary or partial) differential equation is to be determined so that the solution satisfies an overdetermined set of boundary conditions. We shall consider the numerical solution of such problems for both similar and non-similar two dimensional laminar flows. In the former case our technique can also be used to obtain the reverse-flow solutions of the Falkner–Skan

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equation. There is no difficulty in adopting our methods to turbulent flows (using an eddy viscosity formulation) but we do not include such calculations here. The extension to three dimensional boundary-layer flows will be reported elsewhere.

In brief, our procedure is to treat the unknown pressure distribution as an "eigenvalue" which is approximated by a Newton iteration scheme based on satisfying the excess boundary condition. It turns out that for each iteration a standard boundary-layer flow problem must be solved. Thus a key element in the present work is a very accurate and efficient difference scheme [1–3] for computing similar and nonsimilar boundary-layer flows (in which the pressure distribution is given). This nonlinear eigenvalue approach has previously been used [4] to get reverse-flow solutions of the Falkner–Skan equation by means of shooting techniques. Even for that problem, however, the present finite difference method seems superior.

2. The Inverse Problem and Newton's Method

For incompressible laminar flows over a plane surface the boundary-layer equations can be reduced to the dimensionless form [2]:

$$\frac{\partial^3 f}{\partial \eta^3} + f\frac{\partial^2 f}{\partial \eta^2} + \beta(\xi)\left[1 - \left(\frac{\partial f}{\partial \eta}\right)^2\right] = 2\xi\left[\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \xi\,\partial \eta} - \frac{\partial^2 f}{\partial \eta^2}\frac{\partial f}{\partial \xi}\right]. \tag{1}$$

Here $f(\xi, \eta)$ is proportional to a stream function, $\xi \geqslant 0$ is a transformed streamwise variable, $\eta \geqslant 0$ measures distance in the boundary layer and $\beta(\xi)$ is the pressure-gradient parameter. Specifically $\beta(\xi) \equiv (2\xi/u_e)\, du_e/d\xi$, where $u_e(x)$ is the external velocity field which is usually assumed to be known. The most general boundary conditions are of the form

$$\left.\begin{aligned} &\text{(a)} \qquad f(\xi, 0) = f_w(\xi); \qquad \frac{\partial f}{\partial \eta}(\xi, 0) = u_w(\xi) \\ &\text{(b)} \qquad \frac{\partial f}{\partial \eta}(\xi, \eta_\infty) = 1 \end{aligned}\right\} \xi \geqslant 0. \tag{2}$$

Here $f_w(\xi) \not\equiv 0$ allows us to simulate mass transfer at the wall, $u_w(\xi) \not\equiv 0$ allows us to simulate a moving wall and $\eta_\infty = \eta_\infty(\xi)$ is the outer edge of the boundary layer where the external velocity field is attained. The problem (1), (2) with $\beta(\xi)$ specified is the typical nonsimilar plane laminar boundary-layer problem; for brevity we shall call it the *standard problem*. It is easy to formulate this problem so that $\eta_\infty(\xi)$ is also to be determined. For simplicity we shall not include these modifications here and in fact we shall take $\eta_\infty(\xi) \equiv$ const throughout the present work.

The inverse problem results from requiring that the wall shear be specified, that is,

$$(\partial^2 f/\partial\eta^2)(\xi, 0) = S(\xi), \qquad \xi \geqslant 0. \tag{3}$$

The problem (1)–(3) is in general overdetermined and thus we cannot specify $\beta(\xi)$ arbitrarily. Rather we must determine $\beta(\xi)$ as well as $f(\xi, \eta)$ to solve this problem. We shall do this by considering $\beta(\xi)$ to be an "eigenvalue" and determine it so that (3) is satisfied.

More precisely, let the solution of the standard problem, (1), (2) with $\beta(\xi)$ specified, be denoted by

$$f(\xi, \eta) = \mathbb{F}(\xi, \eta, \beta(\xi)) \tag{4}$$

[Technically $\mathbb{F}$ is a nonlinear operator mapping an appropriate class of pressure gradients, $\beta(\xi)$, into solutions, $f(\xi, \eta)$, of (1), (2). However, we proceed formally to derive our solution procedure and numerical methods. The nonlinear functional analysis required to make our considerations rigorous has not, to our knowledge, been carried out. It is unlikely that such an analysis would aid in devising better numerical schemes. But of course it would be of great interest for other reasons.] Using the solution, or rather solution operator (4), we form

$$\phi(\beta(\xi)) \equiv \frac{\partial^2 \mathbb{F}(\xi, 0; \beta(\xi))}{\partial\eta^2} - S(\xi), \qquad \xi \geqslant 0 \tag{5}$$

and seek $\beta(\xi)$ such that $\phi(\beta(\xi)) = 0$ on $\xi \geqslant 0$. Clearly if $\beta = \beta^*(\xi)$ is a "root" of this nonlinear operator equation then $f^*(\xi, \eta) \equiv \mathbb{F}(\xi, \eta; \beta^*(\xi))$ is a solution of (1)–(3).

To solve $\phi(\beta(\xi)) = 0$ we employ Newton's method. Thus with some estimate $\beta^{(0)}(\xi)$ of the desired pressure gradient we define the sequence $\{\beta^{(\nu)}(\xi)\}$ by setting

$$\beta^{(\nu+1)}(\xi) \equiv \beta^{(\nu)}(\xi) + \delta^{(\nu)}(\xi), \tag{6}$$

using this in $\phi(\beta^{(\nu+1)}(\xi)) = 0$ and retaining at most linear terms in $\delta^{(\nu)}(\xi)$. Recalling (5) this linearization procedure yields

$$\frac{\partial^3 \mathbb{F}(\xi, 0, \beta^{(\nu)}(\xi))}{\partial\eta^2\, \partial\beta}\, \delta^{(\nu)}(\xi) = S(\xi) - \frac{\partial^2 \mathbb{F}(\xi, 0, \beta^{(\nu)}(\xi))}{\partial\eta^2}, \qquad \xi \geqslant 0. \tag{7}$$

[This is now a linear operator equation for the determination of $\delta^{(\nu)}(\xi)$. Indeed technically $\partial/\partial\beta$ used in the above Taylor expansion represents Frechet differentiation of the operator $\partial^2\mathbb{F}/\partial\eta^2$. But again we avoid any attempt at rigor and proceed formally.] Upon solving (7) for $\delta^{(\nu)}(\xi)$ we form $\beta^{(\nu+1)}(\xi)$ as in (6) then use this in (1), (2) whose solution is now $\mathbb{F}(\xi, \eta, \beta^{(\nu+1)}(\xi))$. One cycle of the iteration scheme is thus completed. The iterates will converge, say $\beta^{(\nu)} \to \beta^*(\xi)$, if the initial guess $\beta^{(0)}(\xi)$ is appropriately chosen. In fact, the convergence is generally expected to be quadratic; that is $\| \delta^{(\nu+1)} \| = O(\| \delta^{(\nu)} \|^2)$ for ν sufficiently large.

The above procedure for solving the inverse problem for nonsimilar (two-dimensional) flows is easily specialized to solve similar (one-dimensional) flows. Indeed we simply set $\xi = 0$ and (1) reduces to the Falkner–Skan equation. Then (1)–(3) with $\xi = 0$ is precisely a nonlinear eigenvalue problem for the scalar eigenvalue $\beta = \beta(0)$ and the function of one variable $f(\eta) \equiv f(0, \eta)$. The indicated iteration scheme (6), (7) is just Newton's method for solving the scalar equation $\partial^2 \mathbb{F}(0, 0; \beta(0))/\partial \eta^2 = S(0)$. The numerical method and even the computer program are easily specialized to solve this problem. In fact in order to solve nonsimilar flow problems we first solve the corresponding similar flow problem which results from setting $\xi = 0$. In this way we determine the "initial" data $f(0, \eta)$ and $\beta(0)$ which are required for the general numerical scheme for nonsimilar flows.

3. Numerical Procedures; the Box Scheme

On the strip $\{\xi \geqslant 0, 0 \leqslant \eta \leqslant \eta_\infty\}$ we place an arbitrary rectangular net of points $\{\xi_n, \eta_j)$ with

$$\begin{aligned} &\text{(a)} \qquad \xi_0 = 0, \qquad \xi_n = \xi_{n-1} + k_n, \qquad n = 1, 2, \ldots, \\ &\text{(b)} \qquad \eta_0 = 0, \qquad \eta_j = \eta_{j-1} + h_j, \qquad 1 \leqslant j \leqslant J, \qquad \eta_J = \eta_\infty. \end{aligned} \tag{8}$$

No additional restrictions are placed on the meshwidths h_j and k_n. The partial differential Eq. (1) is now written as a first-order system. This is crucial for the application of the Box Scheme [1]. We recall that the conservation equations from which (1) was derived were essentially first order to begin with (we need only consider the strain normal to the body as a basic variable). These original equations would be preferable if not for the fact that the scaling introduced to get (1) eliminates singularities which occur at the leading edge. Thus we replace (1) by

$$\begin{aligned} &\text{(a)} \qquad \partial f/\partial \eta = u, \\ &\text{(b)} \qquad \partial u/\partial \eta = v, \\ &\text{(c)} \qquad \frac{\partial v}{\partial \eta} = \beta(\xi)[u^2 - 1] - fv + 2\xi \left[u \frac{\partial u}{\partial \xi} - v \frac{\partial f}{\partial \xi} \right]. \end{aligned} \tag{9}$$

The boundary conditions (2) become

$$\left.\begin{aligned} &\text{(a)} \qquad f(\xi, 0) = f_w(\xi), \qquad u(\xi, 0) = u_w(\xi) \\ &\text{(b)} \qquad u(\xi, \eta_\infty) = 1 \end{aligned}\right\} \ \xi \geqslant 0 \tag{10}$$

and the additional constraint (3) is simply

$$v(\xi, 0) = S(\xi), \qquad \xi \geqslant 0. \tag{11}$$

The difference approximations to (9) are obtained by considering one box or mesh rectangle, $R_{nj} \equiv \{\xi_{n-1} \leqslant \xi \leqslant \xi_n, \eta_{j-1} \leqslant \eta \leqslant \eta_j\}$, and making the best possible approximations at any points in R_{nj} using at most values of (f, u, v) at the four corners. In particular, if (f_j^n, u_j^n, v_j^n) are to approximate (f, u, v) at (ξ_n, η_j) the above suggests as possible difference equations:

$$\begin{aligned}
&\text{(a)} \quad h_j^{-1}(f_j^n - f_{j-1}^n) = u_{j-1/2}^n, \\
&\text{(b)} \quad h_j^{-1}(u_j^n - u_{j-1}^n) = v_{j-1/2}^n, \\
&\text{(c)} \quad h_j^{-1}(v_j^n - v_{j-1}^n) = \beta_{n-1/2}[(u^2)_{j-1/2}^{n-1/2} - 1] - (fv)_{j-1/2}^{n-1/2} \\
&\qquad\qquad + 2\xi_{n-1/2}k_n^{-1}[u^{n-1/2}(u^n - u^{n-1}) - v^{n-1/2}(f^n - f^{n-1})]_{j-1/2}.
\end{aligned} \tag{12}$$

Here we have introduced a notation for averages and intermediate values as

$$\begin{aligned}
\xi_{n-1/2} &\equiv \xi_n - k_n/2, \qquad & \beta_{n-1/2} &\equiv (\beta_n + \beta_{n-1})/2, \\
v_{j-1/2}^n &\equiv (v_j^n + v_{j-1}^n)/2, \qquad & u_j^{n-1/2} &\equiv (u_j^n + u_j^{n-1})/2,
\end{aligned}$$

$$(fv)_{j-1/2}^{n-1/2} \equiv (f_j^n v_j^n + f_j^{n-1} v_j^{n-1} + f_{j-1}^n v_{j-1}^n + f_{j-1}^{n-1} v_{j-1}^{n-1})/4, \quad \text{etc.}$$

In (12a,b) we have centered the difference approximations to (9a,b) at $(\xi_n, \eta_{j-1/2})$ and in (12c) the approximation to (9c) are centered at $(\xi_{n-1/2}, \eta_{j-1/2})$, the mid-point of R_{nj}. These difference equations are extremely compact, have second-order accuracy and allow arbitrary nonuniform nets. The forms used for the nonlinear terms in (12c) are not uniquely determined but only require that symmetric centering is maintained. Thus one could use in place of $(fv)_{j-1/2}^{n-1/2}$ the product of averages $(f_{j-1/2}^{n-1/2} v_{j-1/2}^{n-1/2})$, etc. We advocate a choice which minimizes the computations.

3.1. *Solution of Standard Problems*

If we assume that the β_n are known and that $(f_j^{n-1}, u_j^{n-1}, v_j^{n-1})$ are known for all $0 \leqslant j \leqslant J$ then (12) for $1 \leqslant j \leqslant J$ and the boundary conditions (10) yield a nonlinear algebraic system of $3J + 3$ equations in as many unknowns (f_j^n, u_j^n, v_j^n). This system can be solved very effectively by using Newton's method. The details are presented in [2, 3] so we do not repeat them here. The important observation is that the linearized equations obtained by applying Newton's method to (10) and (12) form a block tridiagonal system (with 3×3 blocks) and this system can be solved in a very efficient manner. To start this procedure solutions are obtained for $n = 0$, that is on $\xi_0 = 0$ by using slight but obvious modifications in (12c) to get a very efficient difference scheme for the Falkner–Skan equation. The initial estimate of the solution used in the iteration scheme at $\xi = \xi_n$ is the previously converged solution at $\xi = \xi_{n-1}$. Thus for all downstream stations but $\xi = \xi_0$ we have reasonably accurate initial guesses and Newton's method converges rapidly (in 1–3 iterations for most applications, see [2]). At the intial station $\xi_0 = 0$, we must work a bit more since such good initial estimates are not available.

Above we have outlined our numerical method for computing standard non-similar boundary-layer flows. For a given $\beta(\xi)$, this method can be used to accurately approximate $\mathbb{F}(\xi, \eta; \beta(\xi))$ in (4). There are three sources for the errors in these approximations: (i) rounding off in finite precision arithmetic; (ii) iteration errors due to terminating the Newton iterates at a finite stage; (iii) truncation errors due to finite (nonzero) mesh steps. The magnitudes of these errors will in part dictate how we use the scheme to solve the inverse problems.

3.2. *Solution of Inverse Problems*

Suppose now that the inverse problem has been solved (or accurately approximated) for $\xi \leqslant \xi_{n-1}$. Thus we assume known $(f_j^{n-1}, u_j^{n-1}, v_j^{n-1})$ for $0 \leqslant j \leqslant J$ as well as $\beta_{n-1} = \beta(\xi_{n-1})$. We must compute β_n and (f_j^n, u_j^n, v_j^n) for $0 \leqslant j \leqslant J$ to satisfy (10)–(12). To do this we use the nonlinear eigenvalue approach indicated in section 2. Thus if for a fixed $\beta_n = \beta$, say, and $\xi = \xi_n$ the solution of (10) and (12) is denoted by $(f_j^n(\beta), u_j^n(\beta), v_j^n(\beta))$, then (11) becomes

$$v_0^n(\beta) = S(\xi_n). \tag{13}$$

This is solved by Newton's method, iterating on β, that is,

$$\beta^{(\nu+1)} = \beta^{(\nu)} - [v_0^n(\beta^{(\nu)}) - S(\xi_n)]/[\partial v_0^n(\beta^{(\nu)})/\partial\beta]. \tag{14}$$

We call this the "outer" iteration. The "inner" iteration is the previous Newton procedure employed to solve (10), (12) for a fixed given value of $\beta = \beta^{(\nu)}$, say. In principle, each outer iteration could require several inner iterations.

To use Newton's method as in (14) we must evaluate $\partial v_0^n(\beta)/\partial\beta$. This can be approximated in either of two ways. First by differentiating with respect to $\beta = \beta_n$ in (10) and (12), recalling that $(f_j^{n-1}, u_j^{n-1}, v_j^{n-1})$ are independent of β_n, yields a linear system of difference equations for the quantities $(\partial/\partial\beta)(f_j^n(\beta), u_j^n(\beta), v_j^n(\beta))$. The second way is to differentiate with respect to $\beta = \beta(\xi)$ in (9) to get, in terms of

$$F(\xi, \eta, \beta) \equiv \frac{\partial f(\xi, \eta, \beta)}{\partial\beta}, \qquad U(\xi, \eta, \beta) \equiv \frac{\partial u(\xi, \eta, \beta)}{\partial\beta}, \qquad V(\xi, \eta, \beta) \equiv \frac{v(\xi, \eta, \beta)}{\partial\beta},$$

the variational equations:

$$
\begin{aligned}
&\text{(a)} \qquad \partial F/\partial \eta = U, \\
&\text{(b)} \qquad \partial U/\partial \eta = V, \\
&\text{(c)} \qquad \partial V/\partial \eta = \beta\, 2uU - fV - vF \\
&\qquad\qquad + 2\xi \left[u \frac{\partial U}{\partial \xi} + U \frac{\partial u}{\partial \xi} - v \frac{\partial F}{\partial \xi} - V \frac{\partial f}{\partial \xi} \right] + (u^2 - 1).
\end{aligned} \tag{15}
$$

Boundary conditions are, from differentiation in (10),

$$
\begin{aligned}
&\text{(a)} \qquad F(\xi, 0, \beta) = 0, \qquad U(\xi, 0, \beta) = 0, \\
&\text{(b)} \qquad U(\xi, \eta_\infty, \beta) = 0.
\end{aligned} \tag{16}
$$

Using the Box Scheme on the net (8) we replace (15) by difference equations for $(F_j^n(\beta), U_j^n(\beta), V_j^n(\beta))$. Both of the indicated methods yield block tridiagonal linear systems which are easily solved by the previously mentioned factorization procedure. In our calculations we have used the latter method. It corresponds more closely to the procedure of Section 2 evaluated approximately by finite differences.

In addition to the sources of error (i)–(iii) in solving the standard problem (see end of Section 3.1) there is, for the inverse problem, (iv) an iteration error due to terminating the Newton iterates used to solve (13). A well-balanced solution procedure should maintain these error terms of about the same magnitude. In fact the roundoff errors (i), are with no additional difficulty usually several orders of magnitude less than the other errors. The first Newton iteration errors, (ii), are with only *one iteration* of the same order as the truncation errors provided that (a) we are not at the initial point $\xi_0 = 0$, and (b) the solution at the previous point had balanced errors and was used as the intial estimate. For this reason we need use only one inner iteration (as in Section 3.1) for each outer iteration (as in Section 3.2) for all $\xi_n > 0$. With this procedure the entire scheme, that is essentially the outer iterations, were observed to converge quadratically. At the initial point ξ_0, the inner iterations were repeated till convergence for the first outer iteration and afterward only one inner iteration was used for each outer iteration. This is also the procedure that is used for similar flows, in which our problem reduces to the Falkner–Skan equation, or, equivalently, to the case with $\xi_0 = 0$.

4. Results for Similar Flows; Falkner–Skan

For similar flows we simply use the Falkner–Skan equations obtained by setting $\xi = 0$ in (1). Computations were done for both positive and negative wall shear. In the former case we used $\eta_\infty = 6$ and $\Delta\eta = 0.1$. To start the calculations for the value $f''(0) = S = 0.46960$ we used the initial estimate $\beta^{(0)} = 0$ and the quantities $f^{(0)}(\eta)$, $u^{(0)}(\eta)$, $v^{(0)}(\eta)$ were obtained from the Pohlhausen type of velocity profile, with $\zeta \equiv \eta/\eta_\infty$:

$$df^{(0)}(\eta)/d\eta \equiv u^{(0)}(\eta) \equiv \zeta\{2(1-\zeta^2) + \zeta^3 + (\beta\eta_\infty{}^2/6)[1 - 3\zeta(1-\zeta) - \zeta^3]\}.$$

When convergence was obtained for a given value of $f''(0)$, the converged values were used as the initial estimate for the next, slightly different, value of $f''(0)$. In Table I, we present the computed values of β for nine given values of $f''(0) \geqslant 0$ and compare the results with those of Smith [5]. As a consistency check, when a converged value for β was obtained the standard problem was solved using this

TABLE I

Positive wall-shear solutions for Falkner–Skan flows

Number of outer iterations (ν)	$f''(0)$		β (computed)	
	Given	Computed	Present method	Smith
2	0.46960	0.46960	−0.00031	0
3	0.40032	0.40032	−0.05031	−0.05
3	0.31927	0.31926	−0.10017	−0.10
3	0.23974	0.23971	−0.14024	−0.14
3	0.19078	0.19077	−0.16016	−0.16
3	0.12864	0.12880	−0.18025	−0.18
3	0.08570	0.08553	−0.19023	−0.190
3	0.05517	0.05490	−0.19528	−0.195
2	0	−0.00483	−0.20259	−0.198834

value of β to compute the wall shear. The resulting computed values of $f''(0)$ are also listed in the table. The number of outer iterations for convergence, which required that $|\beta^{(\nu+1)} - \beta^{(\nu)}| \leqslant 10^{-4}$, is indicated. Quadratic convergence was apparent in all cases.

In the computations for negative wall shear (reverse flows) we take $\eta_\infty = 10$ and $\Delta\eta = 0.1$. Starting with $f''(0) = S = -0.097$ we use $\beta^{(0)} = -0.18$ and initial profiles obtained from

$$\frac{df^{(0)}(\eta)}{d} \equiv u^{(0)}(\eta) = -\frac{\zeta}{(1-\zeta_0)^2}\{\zeta_0(3-2\zeta_0) - (3-\zeta_0{}^2)\,\zeta - (\zeta_0 - 2)\,\zeta^2\},$$
$$\zeta = \eta/\eta_\infty \qquad \zeta_0 = \eta_0/\eta_\infty$$

with $\zeta_0 = 4.5$. Again when convergence is obtained the results are used to furnish the initial estimates for the next nearby value of $f''(0)$. In Table II we present results for eight given values of $f''(0) < 0$ and compare with the previous calculations of Cebeci and Keller [4] and of Stewartson [6]. The computed $f''(0)$ and number of iterations are as described for Table I. The convergence for negative wall shear cases was not observed to be quadratic over most of the iterations. This is reflected in the relatively large number of iterations required for convergence. Better initial guesses would remedy this and could have been obtained by taking smaller increments in $f''(0)$.

TABLE II

Comparison of reverse-flow solutions for Falkner–Skan flows

Number of outer iterations (ν)	$f''(0)$		β (computed)		
	Given	Computed	Present method	Stewartson	Cebeci–Keller
2	−0.097	−0.09701	−0.181428	−0.18	−0.180553
6	−0.132	−0.13203	−0.154162	−0.15	−0.152118
6	−0.141	−0.14103	−0.135446		
10	−0.132	−0.13199	−0.078662		−0.079596
6	−0.108	−0.10799	−0.049735	−0.05	−0.049745
5	−0.097	−0.09698	−0.040014		−0.040286
4	−0.074	−0.07388	−0.024834	−0.025	−0.024789
7	−0.040	−0.04000	−0.009060		−0.009162

5. Results for Nonsimilar Flows

For nonsimilar flows we consider two distributions of wall shear, $f''(\xi, 0) \equiv S(\xi)$, given by

$$\begin{aligned} &\text{Case A:} \qquad S(\xi) = 0.4696(1 - \xi), \\ &\text{Case B:} \qquad S(\xi) = 1.232588(1 - \xi). \end{aligned} \tag{17}$$

At $\xi = 0$, Case A corresponds to a flat plate flow (that is zero pressure gradient, $\beta = 0$, as is seen from Table I) and Case B corresponds to a stagnation point flow (that is $\beta = 1$, see for instance Smith [5]). We have chosen shear variations that vanish at $\xi = 1$. This is a flow separation point and the validity of the boundary layer approximations are in doubt in the neighborhood of such points. A severe test of any numerical method is to see how close to separation one can compute.

In both cases we compute with $\Delta\xi = 0.05$, $\eta_\infty = 6$ and two different uniform $\Delta\eta$-meshes: $h^{(0)} = 0.5$ and $h^{(1)} = 0.25$. The convergence test for the outer iterations was $|\beta^{(\nu+1)}(\xi_n) - \beta^{(\nu)}(\xi_n)| \leqslant 10^{-4}$ Quadratic convergence was observed everywhere except at $\xi_{19} = 0.95$ for Case A and at $\xi_{20} = 1.00$ for Case B where the iterations diverged.

TABLE III

Computed pressure-gradient parameter β as a function of ξ for Case A

ξ	Given $f''(\xi, 0)$	Computed $f''(\xi, 0)$ $h^{(0)} = 0.5$	Computed $f''(\xi, 0)$ $h^{(1)} = 0.25$	Computed $f''(\xi, 0)$ $h^{(0)}, h^{(1)}$	Computed $(-\beta)$ $h^{(0)} = 0.5$	Computed $(-\beta)$ $h^{(1)} = 0.25$	Computed $(-\beta)$ $h^{(0)}, h^{(1)}$
0	0.46960	0.46950	0.46955	0.46957	0.00708	0.00179	0.00003
0.10	0.42264	0.42270	0.42265	0.42264	0.04976	0.04532	0.04383
0.20	0.37568	0.37566	0.37571	0.37572	0.08939	0.08553	0.09068
0.30	0.32872	0.32866	0.32870	0.32871	0.12562	0.12225	0.12113
0.40	0.28176	0.28182	0.28170	0.28166	0.15835	0.15522	0.15018
0.50	0.23480	0.23480	0.23480	0.23480	0.18709	0.18408	0.18308
0.60	0.18784	0.18778	0.18784	0.18786	0.21139	0.20845	0.20747
0.70	0.14088	0.14095	0.14089	0.14086	0.23051	0.22761	0.22664
0.80	0.09392	0.09397	0.09396	0.09395	0.24347	0.24041	0.23940
0.90	0.04696	0.04706	0.04704	0.04703	0.24790	0.24479	0.24376

TABLE IV

Computed pressure-gradient parameter β as a function of ξ for Case B

ξ	Given $f''(\xi, 0)$	Computed $f''(\xi, 0)$ $h^{(0)} = 0.5$	Computed $f''(\xi, 0)$ $h^{(1)} = 0.25$	Computed $f''(\xi, 0)$ $h^{(0)}, h^{(1)}$	Computed (β) $h^{(0)} = 0.5$	Computed (β) $h^{(1)} = 0.25$	Computed (β) $h^{(0)}, h^{(1)}$
0	1.23259	1.23255	1.23259	1.23260	0.95633	0.98862	0.99938
0.10	1.10933	1.10934	1.10935	1.10935	0.73074	0.75308	0.76053
0.20	0.98607	0.98612	0.98608	0.98607	0.52293	0.53733	0.54213
0.30	0.86281	0.86286	0.86283	0.86282	0.33321	0.34164	0.34445
0.40	0.73955	0.73959	0.73957	0.73956	0.16221	0.16634	0.16772
0.50	0.61629	0.61631	0.61630	0.61630	0.01068	0.01203	0.01248
0.60	0.49304	0.49301	0.49306	0.49307	−0.12016	−0.12017	−0.12017
0.70	0.36978	0.36976	0.36976	0.36976	−0.22816	−0.22837	−0.22844
0.80	0.24652	0.24651	0.24649	0.24648	−0.30942	−0.30877	−0.30856
0.90	0.12326	0.12323	0.12326	0.12327	−0.35514	−0.35302	−0.35232
0.95	0.06163	0.06159	0.06173	0.06177	−0.35787	−0.45517	−0.35427

The results of the calculations are summarized in Tables III and IV. Again to check consistency the computed $\beta(\xi)$ were employed in the standard problem to compute the resulting stress, $f''(\xi, 0)$. All calculations were repeated on two η-nets and the results extrapolated, via Richardson extrapolation as described in [1], to obtain higher order accuracy. The "improvements" thus obtained were not as pronounced as were those in [2, 3]. The reason for this is that only one inner iteration was performed for each outer iteration and thus the iteration and truncation errors were of the same order of magnitude. The extrapolation procedure can only reduce the truncation errors. Thus in order for it to be effective the iteration errors must be much less than the truncation errors. Thus we should perform at least two (quadratically converging) inner iterations per outer iteration if one Richardson extrapolation is to be done.

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